Finite Maturity Optimal Stopping of Lévy Processes with Running Cost, Stopping Cost and Terminal Gain

B. A. Surya*
School of Business and Management
Bandung Institute of Technology
Jln. Ganesha No. 10, Bandung 40132, Indonesia.

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Abstract

Motivated by problems in mathematical finance and insurance, this paper discusses optimal stopping problem in general setting. It considers discounted running cost and stopping cost in addition to terminal gain in the objective function, subject to be optimized over finite-time period. The underlying source of uncertainty is modeled by Lévy processes. We derive early exercise premium representation for the value function based on a partial integro-differential free-boundary problem associated with the optimal stopping problem. The representation gives rise to a nonlinear integral equation for the optimal stopping boundary. The integral equation generalizes that of found in Kim [23], Myneni [28], Carr et al. [8], Jacka [19], Pham [32], and Peskir [30]. The boundary can be characterized as a unique solution of the integral equation within the class of continuous decreasing function of time to maturity. We show that the continuity of the boundary holds when the stopping cost function is either time-independent or decreasing in time. Uniqueness of such solution holds when the running cost and stopping cost functions satisfy a differential inequality. By reformulating the free-boundary problem as a linear complementarity, the problem is solved iteratively by adapting the implicit-explicit method of Cont and Voltchkova [10] and the Brennan-Schwartz [7] algorithm that was implemented in Jaillet et al. [21] and Almendral [1] for the pricing of American put option. We give an example in optimal capital structure. We also verify numerically the recent results in Kyprianou and Surya [25] that the smooth pasting condition may not hold for general Lévy processes.

Keywords: Optimal stopping, free-boundary, linear complementarity problem

1 Introduction

Most problems in finance can be formulated in terms of an optimal stopping problem, which constitutes of a certain gain function to be optimized for a given underlying random process of interest. The question centering around this problem is to find within a specified period of time an optimal stopping time at which the objective function is maximized. For an example, as discussed in details by Bensoussan [5], Karatzas and Shreve [22], Myneni [28], Jacka [19] and quite recently by Peskir [30], the arbitrage-free price of the American put option with strike price K coincides with the value function of the optimal stopping problem with terminal gain function $G(x) = (K - x)^+$. The optimal stopping time for this problem is the first time the price process, which is governed by diffusion process, goes below a time-dependent boundary. When the maturity time of the option is finite, the problem is essentially two-dimensional in

*Corresponding email address: budhi.surya@sbm-itb.ac.id.

the sense that it consists of finding the value function and the optimal stopping boundary simultaneously; that is to say that the value function can be seen as a function of an unknown boundary. Therefore, from an analytical point of view, solving the problem is difficult.

The first and one of the most penetrating mathematical analysis of the American put option optimal stopping problem was due to McKean [27]. There the problem was transformed into a *free-boundary problem* for the value function and the boundary. Solving the free boundary problem, McKean obtained the American option price explicitly in terms of the boundary. McKean's work was taken further by van Moerbeke [38]. Motivated by the physical problem of *the condition of heat balance* (i.e., the law of conservation of energy), van Moerbeke [38] introduced a so-called the *smooth pasting* condition to determine the boundary and specify the value function. This condition dictates that the value and the payoff functions must join smoothly at the boundary. The derivation of the smooth pasting condition for diffusion processes are given by Grigelionis and Shiryaev [16], Shiryaev [35], Chernoff [9], McKean [27] and Myneni [28] using Taylor approximation of the value function around the boundary and by Bather [4] and van Moerbeke [38] using Taylor expansion of the payoff function around the boundary plus the assumption that the boundary is *regular*[1] for the interior of the stopping region for the underlying process. Since the value function is not known a priori, the approach of Bather [4] and van Moerbeke [38] is more satisfactory than the others.

As an alternative to the Taylor expansion method, Peskir [30] introduced a probabilistic approach to prove the smooth pasting condition. The main approach is based on a *change of variable formula with local time-space* on curves, see [29] for further details. This formula extends further the Itô-Tanaka formula for convex functions (see for instance Revusz and Yor [34]). Using the change of variable formula and the free boundary problem, Peskir [30] derived the smooth pasting condition. (See also Peskir and Shiryaev [31] for more discussion on recent development of local time-space calculus in the theory of optimal stopping.)

Based on the free-boundary formulation of the American put option optimal stopping problem, van Moerbeke [38], Myneni [28], El Karoui and Karatzas [14], Jacka [19], Carr et al. [8], Pham [32] and later Peskir [30] showed using Itô-Doob-Meyer decomposition of the value function into martingale and potential processes that the optimal boundary can be characterized as a solution to a nonlinear integral equation. Such an equation was obtained earlier by Friedman [15] in 1959 for a one-dimensional free-boundary problem of ice melting. This nonlinear integral equation for the boundary is known as the *Riesz decomposition* for the value function and has a clear economical meaning to the *early exercise premium* representation of the value function. We refer among others to Kim [23], Myneni [28] and Carr et al. [8].

The existence and local uniqueness of a solution to the nonlinear integral equation for the boundary was proved by Friedman [15] and van Moerbeke [38] using the fixed point theorem (contraction principle) first for a small time interval and extending it to any interval of time using induction arguments. The result of applying the fixed point theorem is that the nonlinear equation involves continuous differentiability of the curve boundary, a condition that is needed to be proved a priori, and results in a long computation and strong condition imposed on the boundary. In contrast to the fixed point method, Jacka [19] and later Peskir [30] introduced a probabilistic approach to prove the existence and uniqueness of a solution to the nonlinear integral equation. The key ingredient of the proof is based on the smooth pasting condition and, in particular, the Itô-Doob-Meyer decomposition of the value function. (Note that the Itô-Doob-Meyer decomposition underlies the basic principle of the theory of optimal stopping developed earlier by Snell [36], Dynkin [12] and Dynkin and Yushkevich [13].) However, the incorporation of the smooth pasting condition in the proof was made clear by Peskir [30] using a change-of-variable formula with local time on curve boundary developed in [30].

This paper attempts to generalize the American put option optimal stopping problem. The generalization considers running cost and stopping cost functions in addition to terminal gain

[1] Starting at the boundary, the underlying process makes an immediate visit to the interior points of the stopping region.

in the objective function, subject to be optimized over finite-time period. The source of uncertainty in the underlying price process is governed by Lévy processes. In certain applications such as in the optimal capital structure problem of an equity maximizing firm, the running cost may be considered as the total cost incurred up to the time the firm declares bankruptcy or at terminal time, which ever is sooner, whereas stopping cost describes the cost the firm must settle upon default. See, among others, Leland and Toft [26], Hilberink and Rogers [18], Kyprianou and Surya [25] and Surya and Yamazaki [37] for further details. We formulate a partial integro-differential free-boundary problem that the value function must solve. The free-boundary imposes, among other boundary conditions, the majorant property condition. The latter requires that the value function is always dominant over stopping cost function.

We show using Itô-Doob-Meyer decomposition of the value function that solution to the free-boundary problem coincides with the value function, and the optimal stopping time is characterized as the first time the price process goes below the boundary. We derive Riesz decomposition for the value function. The decomposition gives rise to a nonlinear integral equation for the boundary. Such integral equation generalizes the earlier result of Kim [23], Myneni [28], Carr et al. [8], Jacka [19], Peskir [30], and Pham [32]. We show that the boundary can be characterized as a unique solution to the integral equation within the class of continuous decreasing function of time to maturity. Furthermore, we show that the continuity of the boundary holds when the stopping function is either time-independent or continuous decreasing function of time to maturity and the uniqueness of such solution holds when the running cost and stopping cost functions satisfy a differential inequality. As a result, the value function and the boundary are identified as the unique pair solutions to the free-boundary problem.

Apart from showing uniqueness of solution to the integral equation (and the free-boundary problem), we propose a numerical scheme for the computation of the value function and the boundary. The scheme is based on reformulation of the free-boundary problem as a linear complementarity problem. The latter is solved by adapting the implicit-explicit method of Cont and Voltchkova [10] and the Brennan-Schwartz [7] algorithm, which was implemented by Jaillet et al. [21] and Almendral [1] for pricing American put option. We give an example in optimal capital structure problem and verify numerically the recent results in Kyprianou and Surya [25] that the smooth pasting condition may not hold for general Lévy processes.

The rest of this paper is organized as follows. Section 2 discusses formulation of the optimal stopping problem of interest and the Lévy process as the source of underlying uncertainty. Free-boundary problem associated with the optimal stopping problem is outlined in Section 3. This section contains the main results of this paper. Section 4 is concerned with numerical computation of the free-boundary problem by reformulating the problem in terms of linear complementarity problem. The latter is solved iteratively by applying the implicit-explicit method of Cont and Voltchkova [10] and the Brennan-Schwartz algorithm [7], [21], [1]. Section 5 discusses some numerical examples. We postpone all the proof of main results in the Appendix.

2 Optimal stopping problem formulation

As for the source of underlying uncertainty in the price process, we assume as usual that the uncertainty is generated by a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\mathbb{F} = \{\mathcal{F}_t, 0 \leq t \leq T\}$ satisfying the usual conditions. On this probability space are defined a standard Brownian motion B and a homogeneous Poisson random measure $\nu(dy, dt)$ on $\mathbb{R} \times [0, T]$ counting the number of jumps of size y in the log price at time t. We refer to Jacod and Shiryaev [20] for further details on random measure and its characteristic. We assume that the Poisson random measure ν has the intensity measure $\mu(dy, dt)$ taking the following form

$$\mu(dy, dt) = \Pi(dy)dt, \tag{2.1}$$

where Π is the Lévy measure on $\mathbb{R}\backslash\{0\}$ satisfying the integrability condition

$$\int_{\mathbb{R}} (1 \wedge |y|)\Pi(dy) < \infty. \tag{2.2}$$

This is to say that the jumps in the log price are assumed to have paths of bounded variation. We denote by $\widehat{\nu}$ the compensated jump martingale of the random measure ν defined by

$$\widehat{\nu}(dy, dt) = \nu(dy, dt) - \mu(dy, dt).$$

The dynamics of the price process is governed by the stochastic differential equation

$$\frac{dX_t}{X_{t-}} = (r - \delta + \omega)dt + \sigma dB_t + \int_{\mathbb{R}} \big(e^y - 1\big)\nu(dy, dt), \tag{2.3}$$

where $\sigma \geq 0$ and $r, \delta, \omega \in \mathbb{R}$. In the sequel below we denote by $\mathbb{P}_{t,x}$ the law of X under which it starts at $x \in \mathbb{R}_+$ at time $t \in [0, T]$, and we shall write $\mathbb{E}_{t,x}$ for the expectation operator associated with $\mathbb{P}_{t,x}$. Applying Itô's formula (see e.g., Theorem II.33 in Protter [33] or formula (4.58) in Jacod and Shiryaev [20]) we have for any $s \geq t$, almost surely under $\mathbb{P}_{t,x}$, that

$$X_s = xL_s := xe^{(r-\delta-\frac{\sigma^2}{2}+\omega)s+\sigma B_s+J_s},$$

where $J_t := \sum_{0<s\leq t} \Delta_s$ denotes a pure jump process representing the jump size in the log price and $\{(s, \Delta_s) : s \geq 0\}$ is a Poisson point process on $[0, \infty) \times \mathbb{R}\backslash\{0\}$ with time-space intensity measure $dt \times \Pi(dx)$. See [20] for further details. We will further assume also that

$$\int_{\{|y|>1\}} e^{\beta y}\Pi(dy) < \infty \quad \text{for } \beta = 1, 2. \tag{2.4}$$

The condition (2.4) implies the existence of the first two moments of X. See for instance Theorem 3.6 in Kyprianou [24]. We set the value of the parameter ω to be equal to

$$\omega = \int_{\mathbb{R}} \big(1 - e^y\big)\Pi(dy)$$

to compensate the pure jump process ν in the dynamics of X. By doing this, the discounted price process $\widehat{X} = (e^{-(r-\delta)t}X_t : t \geq 0)$ becomes square-integrable $\mathbb{P}_{t,x}-$martingale:

$$\frac{d\widehat{X}_t}{\widehat{X}_{t-}} = \sigma dB_t + \int_{\mathbb{R}} \big(e^y - 1\big)\widehat{\nu}(dy, dt), \quad \text{and} \quad \sup_{t\in[0,T]} \mathbb{E}\big(\widehat{X}_t^2\big) < \infty.$$

The optimal stopping problem we are interested in is to find the value function

$$\begin{aligned} V(t, x) = \sup_{\tau\in\mathcal{T}_{[t,T]}} \mathbb{E}_{t,x}\Big(&\int_t^{\tau\wedge T} e^{-\int_t^u \alpha(s,X_s)ds}F(u, X_u)du \\ &+ e^{-\int_t^\tau \alpha(s,X_s)ds}G(\tau, X_\tau)\mathbf{1}_{\{\tau<T\}} + e^{-\int_t^T \alpha(s,X_s)ds}H(X_T)\mathbf{1}_{\{\tau\geq T\}}\Big), \end{aligned} \tag{2.5}$$

for some given functions F, G, α and H defined on $\mathbb{R}_+ \times \mathbb{R}_+$ and $\mathbb{R}_+$, respectively. The functions F, G and H represent consecutively the running cost, stopping cost and terminal gain, whereas the function α plays the role as the "discount" factor. We have denoted by $\mathcal{T}_{[t,T]}$ the set of all stopping times taking value in interval $[t, T]$. Note that the function $\alpha(t, x) : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ may not necessarily be interpreted as a discount rate; it could be any positive function.

In full generality we can in principle extend the dynamics of the price process (2.3) to the one in which all the parameters r, δ and σ may be time-space dependent functions; but for simplicity we shall assume throughout this paper that they are just constants.

Due to the early exercise nature of the optimal stopping problem (2.5), we derive *early exercise premium decomposition* of the value function in terms of the function

$$U(t,x) = \mathbb{E}_{t,x}\Big(\int_t^T e^{-\int_t^u \alpha(s,X_s)ds}F(u,X_u)du + e^{-\int_t^T \alpha(s,X_s)ds}H(X_T)\Big). \tag{2.6}$$

The problem (2.6) is discussed in details for multivariate diffusion by Duffie [11] and Heat and Schweizer [17] whom provided a set of verifiable sufficient conditions and practical examples.

Proposition 2.1 (Continuity and majorant property) *Let F, G, α and H be continuous functions on $[0,T]\times[0,\infty)$ and $[0,\infty)$, respectively. Then, the value function V (2.5) is continuous on $[0,T]\times[0,\infty)$ and dominates the stopping cost function G, i.e., $V \geq G$.*

Proof Applying time shift in the integration, the proof follows from the fact that for every $\omega \in \Omega$, the function $(t,x) \to \int_0^{(\tau-t)\wedge(T-t)} \exp\big(-\int_0^{u-t}\alpha(t+s,xL_s)ds\big)F(t+u,xL_u)du + \exp\big(-\int_0^{\tau-t}\alpha(t+s,xL_s)ds\big)G(\tau,xL_\tau)\mathbf{1}_{\{\tau<T\}} + \exp\big(-\int_0^{T-t}\alpha(t+s,xL_s)ds\big)H(xL_T)\mathbf{1}_{\{\tau\geq T\}}$ is continuous on $[0,T]\times[0,\infty)$, hence the value function $V(\cdot,\cdot)$ inherits this property, as taking $\mathbb{P}-$integration over ω and taking the supremum over τ preserves the continuity. The function V dominates the stopping cost function G, because we can always take $\tau \equiv t$ in (2.5). □

3 Free-boundary problem

This section discusses the main results of the paper containing the free-boundary problem formulation for the value function, an integral equation of the boundary, and their uniqueness.

Theorem 3.1 *Suppose that the functions F, G and H of (2.5) are given. Let W be a continuous function on $[0,T]\times[0,\infty)$ and be $C^{1,2}$ on $(0,T)\times(0,\infty)$. Define a curve boundary*

$$b(t) := \inf\{x \in \mathbb{R}_+ : W(t,x) > G(t,x)\}, \tag{3.1}$$

for every $t \geq 0$. Assume that W solves the partial integro-differential equations:

$$-\frac{\partial W}{\partial t}(t,x) - \mathcal{L}W(t,x) = F(t,x), \quad \text{for } t \in (0,T), x > b(t), \tag{3.2}$$

$$W(t,x) = G(t,x), \quad \text{for } t \in (0,T), x \leq b(t), \tag{3.3}$$

$$W(t,x) \geq G(t,x), \quad \text{for } t \in [0,T), x \in \mathbb{R}_+, \tag{3.4}$$

$$-\frac{\partial W}{\partial t}(t,x) - \mathcal{L}W(t,x) \geq F(t,x), \quad \text{for } t \in (0,T), x \in \mathbb{R}_+, \tag{3.5}$$

$$W(T,x) = H(x), \quad \text{for } x \in \mathbb{R}_+, \tag{3.6}$$

where $\mathcal{L}$ is an integro-differential operator defined by

$$\begin{aligned}\mathcal{L}W(t,x) &= \frac{\sigma^2}{2}x^2\frac{\partial^2 W}{\partial x^2}(t,x) + (r-\delta)x\frac{\partial W}{\partial x}(t,x) - \alpha(t,x)W(t,x)\\ &\quad + \int_{\mathbb{R}}\big[W(t,xe^y) - W(t,x) - (e^y-1)x\frac{\partial W}{\partial x}(t,x)\big]\Pi(dy).\end{aligned} \tag{3.7}$$

Then, the function W coincides with the value function V (2.5) for every $(t,x) \in [0,T]\times\mathbb{R}_+$, and the optimal stopping time for (2.5) is the first time the process X (2.3) goes below b, i.e.,

$$\tau_b^- = \inf\{t \leq s \leq T : X_s < b(s)\}, \tag{3.8}$$

and the optimal stopping boundary b solves a nonlinear integral equation

$$\begin{aligned}&\mathbb{E}_{t,b(t)}\Big(e^{-\int_t^T \alpha(s,X_s)ds}H(X_T)+\int_t^T e^{-\int_t^u \alpha(s,X_s)ds}F(u,X_u)du\Big)\\ &=G(t,b(t))-\mathbb{E}_{t,b(t)}\Big(\int_t^T e^{-\int_t^u \alpha(s,X_s)ds}\big[-\frac{\partial G}{\partial t}-\mathcal{L}G-F\big](u,X_u)\mathbf{1}_{\{X_u\leq b(u)\}}du\Big).\end{aligned} \tag{3.9}$$

Observe that the integral equation (3.9) for the optimal boundary generalizes the earlier results which are found in Myneni [28], Jacka [19], Carr et al. [8], Pham [32], and Peskir [30].

Theorem 3.2 *Suppose that the functions G and F in (2.5) are such that*

$$\Big(\frac{\partial G}{\partial t}+\mathcal{L}G+F\Big)(t,x)<0 \quad \textit{for all } t>0 \textit{ and } x\in\mathbb{R}_+. \tag{3.10}$$

Then, the optimal stopping boundary b solves the integral equation (3.9) uniquely.

Corollary 3.3 *The pair (V,b) are the unique solution to (3.2)-(3.7).*

On account of the fact that the value function coincides with the solution to the free-boundary, this would imply that the value function inherits the property of the solution.

As we keep the optimal stopping problem (2.5) as general as possible by not explicitly specifying the property of the functions F, G, H and α, we are unable to characterize the property of the value function $V(t,x)$ in the direction of x. However, the following proposition establishes the property of V as well as the property of the boundary in time direction.

Proposition 3.4 *The value function V satisfies:*

(i) If F, G and H are positive, then $V(t,x)>0$ for every $(t,x)\in(0,T]\times\mathbb{R}_+$.

(ii) $V(\cdot,x)>0$ is decreasing on $[0,T]$ for every $x\in\mathbb{R}_+$.

Proof The first claim in (i) is quite straightforward, due to the positivity of F, G and H imposed. The second claim in (ii) follows from the fact that $\mathcal{T}_{[s,T]}\subseteq\mathcal{T}_{[t,T]}$ for any $s\geq t$. □

Assumption 3.5 *To establish the property of the boundary, we assume that*

(i) the stopping cost function G is continuous on $[0,T]\times[0,\infty)$,

(ii) and $G(\cdot,x)$ is decreasing on $[0,T]$ for every $x\in\mathbb{R}_+$.

Remarks 3.6 *Note that in the absence of running cost function F and G is only time-dependent in (2.5), the requirement (ii) still satisfies the condition (3.10) for the uniqueness of solution to the integral equation (3.9) as well as the free-boundary problem (3.2)-(3.7).*

By adapting the argument of Jacka [19], we establish in the propositions below that the boundary is a continuous decreasing function of time to maturity. To do this, we define:

Definition 3.7 $\mathcal{V}(t,x):=V(T-t,x)$ *and* $\widetilde{b}(t):=b(T-t)$.

Note that the time reverse of the value function preserves the continuity property of the value function $\mathcal{V}$ as well as its majorant property (3.4) over the stopping cost function G.

Proposition 3.8 *The boundary $\widetilde{b}(t)$ is decreasing in t.*

Proof By Proposition 3.4, $\mathcal{V}(\cdot, x)$ is increasing. So, for any $t > 0$; $s \geq 0$ and $\epsilon > 0$

$$\mathcal{V}(t+s, \widetilde{b}(t)+\epsilon) \geq \mathcal{V}(t, \widetilde{b}(t)+\epsilon) > G(t, \widetilde{b}(t)+\epsilon) \geq G(t+s, \widetilde{b}(t)+\epsilon),$$

from which conclude for any $\epsilon > 0$ and $s > 0$ that $\widetilde{b}(t)+\epsilon \geq \widetilde{b}(t+s)$. Thus, $\widetilde{b}(t) \geq \widetilde{b}(t+s)$. □

Proposition 3.9 *The boundary* $\widetilde{b}$ *is left-continuous.*

Proof Since both $\mathcal{V}$ and G are continuous, the set $\mathcal{D} = \{(t,x) : \mathcal{V}(t,x) > G(t,x)\}$ is open, hence $\mathcal{D}^c$ is closed. Next, take a sequence $\{t_n\} \upuparrows t$. We see that $(t_n, \widetilde{b}(t_n)) \in \mathcal{D}^c$ for all n. As a result, $\widetilde{b}(t-) \leq \widetilde{b}(t)$. The left continuity follows from decreasing property of the boundary. □

Jacka [19] proved the continuity of the boundary under diffusion process by taking the advantage of smooth pasting condition at the boundary. Later, Pham [32] extended the proof for jump-diffusion process. Close inspection of the authors' proof suggests that the diffusion term should not be zero, i.e., $\sigma \neq 0$. However, as the price process (2.3) may be governed by pure jump process, the authors' method of proof may not be applicable in this case. Instead, we shall use the majorant property (3.4) and the continuity of $\mathcal{V}(\cdot, x)$ and of $G(\cdot, x)$ on $[0,T]$.

Proposition 3.10 $\widetilde{b}$ *is right-continuous, and, therefore, it is continuous by Proposition 3.9.*

Proof Taking account of (3.4) and the continuity of $\mathcal{V} - G$, we have for any $\epsilon > 0$ that

$$\mathcal{V}(t, \widetilde{b}(t+)+\epsilon) = \mathcal{V}(t+, \widetilde{b}(t+)+\epsilon) > G(t+, \widetilde{b}(t+)+\epsilon) = G(t, \widetilde{b}(t+)+\epsilon),$$

which in turn implies that $\widetilde{b}(t+)+\epsilon \geq \widetilde{b}(t)$ for any $\epsilon > 0$. Hence, we have that $\widetilde{b}(t+) \geq \widetilde{b}(t)$. The right continuity of the boundary follows from the fact that $\widetilde{b}$ is decreasing. □

The following lemma gives the early exercise premium decomposition of the value function.

Lemma 3.1 *Let* G *be continuous on* $[0,T] \times [0,\infty)$ *and be* $C^{1,2}$ *on* $(0,T) \times (0,\infty)$*. Then,*

$$V(t,x) = U(t,x) + \mathbb{E}_{t,x}\Big(\int_t^T e^{-\int_t^u \alpha(s,X_s)ds}\big[-\frac{\partial G}{\partial t} - \mathcal{L}G - F\big](u, X_u)\mathbf{1}_{\{X_u \leq b(u)\}}du\Big),$$

for every $(t,x) \in [0,T] \times \mathbb{R}_+$*, where* U *defined by (2.6) solves the Cauchy problem:*

$$-\frac{\partial U}{\partial t}(t,x) - \mathcal{L}U(t,x) = F(t,x), \quad \textit{for } t \in [0,T), x \in \mathbb{R}_+, \tag{3.11}$$

$$U(T,x) = H(x) \quad \textit{for } x \in \mathbb{R}_+. \tag{3.12}$$

4 Numerical valuation

For the purpose of numerical computation, we will consider the value function V as a function of the log price $z = \log(x)$ and time to maturity $\tau = T - t$. In line to this, we define new function $\mathcal{V}(\tau, z) := V(T-\tau, e^z)$, similarly for F, G, H and α. By doing so, we have that

$$\frac{\partial \mathcal{V}}{\partial \tau} = -\frac{\partial V}{\partial t}, \quad \frac{\partial \mathcal{V}}{\partial z} = x\frac{\partial V}{\partial x}, \quad \frac{\partial^2 \mathcal{V}}{\partial z^2} - \frac{\partial \mathcal{V}}{\partial z} = x^2\frac{\partial^2 V}{\partial x^2},$$

and $\mathcal{V}(\tau, z+y) = V(T-\tau, xe^y)$. By inserting these in (3.2)-(3.7), we then obtain

$$\begin{aligned}
\frac{\partial \mathcal{V}}{\partial \tau}(\tau,z) - \mathcal{A}\mathcal{V}(\tau,z) &= F(\tau,z), \quad \text{for } \tau \in (0,T), z > c(\tau),\\
\mathcal{V}(\tau,z) &= G(\tau,z), \quad \text{for } \tau \in (0,T), z \le c(\tau),\\
\mathcal{V}(\tau,z) &\ge G(\tau,z), \quad \text{for } \tau \in [0,T), z \in \mathbb{R},\\
\frac{\partial \mathcal{V}}{\partial \tau}(\tau,z) - \mathcal{A}\mathcal{V}(\tau,z) &\ge F(\tau,z), \quad \text{for } \tau \in (0,T), z \in \mathbb{R},\\
\mathcal{V}(0,z) &= \quad H(z), \quad \text{for } z \in \mathbb{R},
\end{aligned}$$

where $\mathcal{A}$ is an integro-differential operator of the form

$$\begin{aligned}
\mathcal{A}\mathcal{V}(\tau,z) \; &= \frac{\sigma^2}{2}\frac{\partial^2 \mathcal{V}}{\partial z^2}(\tau,z) + \big(r - \delta - \frac{\sigma^2}{2}\big)\frac{\partial \mathcal{V}}{\partial z}(\tau,z) - \alpha(\tau,z)\mathcal{V}(\tau,z)\\
&+ \int_{\mathbb{R}} \big[\mathcal{V}(\tau,z+y) - \mathcal{V}(\tau,z) - (e^y - 1)\frac{\partial \mathcal{V}}{\partial z}(\tau,z)\big]\Pi(dy).
\end{aligned} \tag{4.1}$$

The boundary $c(\tau)$ is defined by $c(\tau) := \log \widetilde{b}(\tau)$, or equivalently, specified by

$$c(\tau) = \inf\{z \in \mathbb{R} : \mathcal{V}(\tau,z) > G(\tau,z)\}, \quad \tau \in (0,T].$$

Similarly, the boundary value problem (3.11) can be rewritten as

$$\begin{aligned}
\frac{\partial U}{\partial \tau}(\tau,z) - \mathcal{A}U(\tau,z) =& F(\tau,z), \quad \text{for } \; \tau \in [0,T], z \in \mathbb{R},\\
U(0,z) =& H(z), \quad \text{for } \; z \in \mathbb{R}.
\end{aligned} \tag{4.2}$$

Despite the fact that the value function (2.5) uniquely solves the free-boundary problem (3.2)-(3.7), the optimal stopping boundary b is not known a priori, therefore needs to be found as part of solution to the free-boundary problem (3.2)-(3.7). Thus, rather than solving the the free-boundary problem directly, it is more convenient to look at its related Linear Complementarity Problem, for which numerical solution is given in the next section.

$$\begin{aligned}
\frac{\partial \mathcal{V}}{\partial \tau} - \mathcal{A}\mathcal{V} \; &\ge \quad F, \quad \text{for } (\tau,x) \in (0,T) \times \mathbb{R},\\
\mathcal{V} \; &\ge \quad G, \quad \text{for } (\tau,x) \in (0,T] \times \mathbb{R},\\
\Big(\frac{\partial \mathcal{V}}{\partial \tau} - \mathcal{A}\mathcal{V} - F\Big)(\mathcal{V} - G) \; &= \quad 0, \quad \text{for } (\tau,x) \in (0,T) \times \mathbb{R},\\
\mathcal{V}(0,x) \; &= H(x), \quad \text{for } x \in \mathbb{R},
\end{aligned} \tag{4.3}$$

As opposed to the free-boundary problem, the problem (4.3) has removed the dependency on the free-boundary $c(\tau)$, but is left with a set of inequalities. We refer to Bensoussan and Lions [6] for existence (and uniqueness) of solution to the problem. Our aim now is to discretize and give numerical solution to the problem (4.3) which we shall now discuss in the section below.

4.1 Discretization and numerical algorithm

In the discretization of the problem (4.3), we employ the Cont and Voltchkova [10] implicit-explicit method implemented by Almendral [1] for the valuation of American put option. For this purpose, we assume that the Lévy measure $\Pi(dy)$ has the density $\pi(y)$ defined as

$$\pi(y) = \begin{cases} C_1 \frac{e^{-G|y|}}{|y|^{1+Y}}, & \text{if } y < 0,\\ C_2 \frac{e^{-M|y|}}{|y|^{1+Y}}, & \text{if } y > 0, \end{cases} \tag{4.4}$$

The central idea of the discretization is to approximate the partial integro-differential operator $\mathcal{A}$ (4.1) by truncating the integral term close to and away from the origin. In a neighborhood of origin, the integral is approximated by a convection-diffusion operator, that is to say

$$\begin{aligned}
&\int_{\{|y|\leq\epsilon\}}\big(\mathcal{V}(\tau,x+y)-\mathcal{V}(\tau,x)-(e^y-1)\frac{\partial\mathcal{V}}{\partial x}(\tau,x)\big)\pi(y)dy\\
&=\int_{\{|y|\leq\epsilon\}}\big(\mathcal{V}(\tau,x+y)-\mathcal{V}(\tau,x)-y\frac{\partial\mathcal{V}}{\partial x}(\tau,x)-(e^y-1-y)\frac{\partial\mathcal{V}}{\partial x}(\tau,x)\big)\pi(y)dy\\
&\approx\frac{\sigma_\epsilon^2}{2}\frac{\partial^2\mathcal{V}}{\partial x^2}(\tau,x)-\frac{\sigma_\epsilon^2}{2}\frac{\partial\mathcal{V}}{\partial x}(\tau,x),
\end{aligned}$$

where the parameter σ_ϵ^2 is defined by

$$\sigma_\epsilon^2=\int_{\{|y|\leq\epsilon\}}y^2\pi(y)dy.$$

For the integral away from the origin, it is evaluated as follows:

$$\begin{aligned}
&\int_{\{|y|\geq\epsilon\}}\big(\mathcal{V}(\tau,x+y)-\mathcal{V}(\tau,x)-(e^y-1)\frac{\partial\mathcal{V}}{\partial x}(\tau,x)\big)\pi(y)dy\\
&=\big[\mathcal{J}^\epsilon\mathcal{V}\big](\tau,x)-\lambda_\epsilon\mathcal{V}(\tau,x)+\omega_\epsilon\frac{\partial\mathcal{V}}{\partial x}(\tau,x),
\end{aligned}$$

where we have denoted by $\mathcal{J}^\epsilon V$ the convolution term, whereas λ_ϵ and ω_ϵ are defined by

$$\lambda_\epsilon=\int_{\{|y|\geq\epsilon\}}\pi(y)dy\quad\text{and}\quad\omega_\epsilon=\int_{\{|y|\geq\epsilon\}}(1-e^y)\pi(y)dy.$$

Notice that in the case where there are no jumps in the sample paths of the price process X (2.3), the parameters σ_ϵ, λ_ϵ, ω_ϵ and the convolution term $\mathcal{J}^\epsilon\mathcal{V}$ all have zero values.

As explained in [1], the above approximation for the integral has a meaningful probabilistic interpretations: the small jump has been approximated by a Brownian process, whereas the big jumps are approximated by a compound Poisson process. However, as pointed out in [1] that following Asmussen and Rosiński [3] such approximation is valid if and only if $\sigma_\epsilon/\epsilon\to\infty$ as $\epsilon\to 0$, which is only the case when the parameter Y in (4.4) has value in $(0,1)$, the case that the jumps in the price have paths of bounded variation. As a result, the operator $\mathcal{A}$ (4.1) is approximated by $\mathcal{L}^\epsilon:=\mathcal{D}^\epsilon+\mathcal{P}^\epsilon$, where the operators $\mathcal{D}^\epsilon$ and $\mathcal{P}^\epsilon$ are defined by

$$\mathcal{D}^\epsilon\mathcal{V}(\tau,x)\;:=\;\theta_1\frac{\partial^2\mathcal{V}}{\partial x^2}(\tau,x)+\theta_2\frac{\partial\mathcal{V}}{\partial x}(\tau,x)-\alpha(\tau,x)\mathcal{V}(\tau,x),\tag{4.5}$$

$$\mathcal{P}^\epsilon\mathcal{V}(\tau,x)\;=\;\big[\mathcal{J}^\epsilon\mathcal{V}\big](\tau,x)-\lambda_\epsilon\mathcal{V}(\tau,x),\tag{4.6}$$

where the parameters θ_1 and θ_2 are defined by

$$\theta_1=\frac{\sigma^2+\sigma_\epsilon^2}{2}\quad\text{and}\quad\theta_2=r-\delta-\frac{\sigma^2+\sigma_\epsilon^2}{2}+\omega_\epsilon.$$

Subsequently, following [10] and [1], the problems (4.2) and (4.3) are solved by replacing the operator $\mathcal{A}$ (4.1) by $\mathcal{L}^\epsilon$. The time derivative $\partial\mathcal{V}/\partial\tau$ is discretized using the Euler's scheme in time, whereas for the differential operator $\mathcal{L}^\epsilon$ it is discretized by applying the implicit-explicit iteration in space in the following way. Let the time interval $[0,T]$ be divided into N_t equally distant subintervals $(\tau_{j-1},\tau_j]$, with $j=0,1,\dots,N_t$, $\tau_0=0$, $\tau_{N_t}=T$, $\Delta\tau=\tau_j-\tau_{j-1}$ and define $\mathcal{V}^j:=\mathcal{V}(\tau_j,x)$. The implicit-explicit discretization proposed in [10] is given by

$$\frac{\partial\mathcal{V}}{\partial\tau}\;\approx\;\frac{\mathcal{V}^{j+1}-\mathcal{V}^j}{\Delta\tau},\tag{4.7}$$

$$\mathcal{L}^\epsilon\mathcal{V}\;\approx\;\mathcal{D}^\epsilon\mathcal{V}^{j+1}+\mathcal{P}^\epsilon\mathcal{V}^j.\tag{4.8}$$

Notice that the differential part $\mathcal{D}^\epsilon\mathcal{V}$ is treated implicitly, whereas the integral part $\mathcal{P}^\epsilon\mathcal{V}$ is treated explicitly. This discretization, however, imposes a stability restriction on the time step. We refer to [10] for further details. For space discretization, it is done as follows. Define an uniform grid $x_i = x_{\min}+i\Delta x$, $i=0,1,\dots,N_x$, with $\Delta x = (x_{\max}-x_{\min})/N_x$, and $\mathcal{V}_i := \mathcal{V}(\tau, x_i)$. Next, the space derivatives are discretized using the finite difference central schemes:

$$\begin{aligned}\Big(\frac{\partial^2\mathcal{V}}{\partial x^2}\Big)_i &\approx \frac{\mathcal{V}_{i+1}-2\mathcal{V}_i+\mathcal{V}_{i-1}}{(\Delta x)^2},\\ \Big(\frac{\partial\mathcal{V}}{\partial x}\Big)_i &\approx \frac{\mathcal{V}_{i+1}-\mathcal{V}_{i-1}}{2\Delta x}.\end{aligned}$$

Using the above discretization, the vector expressions in (4.8) have the entry form:

$$\begin{aligned}(\mathcal{D}^\epsilon\mathcal{V}^\bullet)_i &= \Big(\frac{\theta_1}{(\Delta x)^2}+\frac{\theta_2}{2\Delta x}\Big)\mathcal{V}^\bullet_{i-1} - \Big(\frac{2\theta_1}{(\Delta x)^2}+\alpha_i^\bullet\Big)\mathcal{V}^\bullet_i + \Big(\frac{\theta_1}{(\Delta x)^2}-\frac{\theta_2}{2\Delta x}\Big)\mathcal{V}^\bullet_{i+1}\\ (\mathcal{P}^\epsilon\mathcal{V}^\bullet)_i &= (\mathcal{J}^\epsilon\mathcal{V}^\bullet)_i - \lambda_\epsilon\mathcal{V}^\bullet_i.\end{aligned}$$

Following [10], [1] and [2], the convolution term $\mathcal{J}^\epsilon V$ is computed using the trapezoidal rule:

$$\begin{aligned}J_i := \big(\mathcal{J}^\epsilon\mathcal{V}\big)_i &= \int_{\{|y|\ge\epsilon\}} \mathcal{V}(\tau, x_i+y)\pi(y)dy\\ &\approx \int_{\{\epsilon\le|y|\le M\epsilon\}} \mathcal{V}(\tau, x_i+y)\pi(y)dy\\ &\approx \epsilon\sum_{m=-M}^{M}\mathcal{V}_{i+m}\pi_m\rho_m, \quad i=0,1,\dots,N_x,\end{aligned}$$

where we have defined $\pi_m := \pi(m\epsilon)$, with $m\neq 0$ and $\pi_0 = 0$, and

$$\rho_m = \begin{cases} 1/2, & \text{if } m\in\{-M,-1,1,M\},\\ 1, & \text{otherwise.}\end{cases}$$

The computation of the sum in evaluating J_i may be slow. To speed it up, we apply fast Fourier transform. We refer to [1] and [2] for further details. Thus, the problem (4.3) becomes

$$\begin{aligned}\frac{\mathcal{V}^{j+1}}{\Delta\tau} - \mathcal{D}^\epsilon\mathcal{V}^{j+1} &\ge d^j := \frac{\mathcal{V}^j}{\Delta\tau} + \mathcal{P}^\epsilon\mathcal{V}^j + F^j,\\ \mathcal{V}^{j+1} &\ge G^{j+1},\\ \Big(\frac{\mathcal{V}^{j+1}}{\Delta\tau} - \mathcal{D}^\epsilon\mathcal{V}^{j+1} - d^j\Big)(\mathcal{V}^{j+1}-G^{j+1}) &= 0,\\ \mathcal{V}^0 &= H.\end{aligned} \tag{4.9}$$

4.2 Discrete linear complementarity problem

In matrix notation, the system of discrete inequalities (4.9) that correspond to the discretization of the linear complementarity problem (4.3) can be rewritten as the following:

$$\begin{aligned}\mathbf{T}_{j+1}\mathcal{V}^{j+1} &\ge d^j,\\ \mathcal{V}^{j+1} &\ge G^{j+1},\\ \big(\mathbf{T}_{j+1}\mathcal{V}^{j+1}-d^j, \mathcal{V}^{j+1}-G^{j+1}\big) &= 0,\\ \mathcal{V}^0 &= H,\end{aligned} \tag{4.10}$$

for $j = 0, 1, \dots, N_t - 1$, where $\mathbf{T}_j$ is a tridiagonal matrix and d^j and G^j are vectors defined by

$$\mathbf{T}_j = \begin{pmatrix} b_1^j & c_1 & & & \\ a_2 & b_2^j & c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & a_{n-1} & b_{n-1}^j & c_{n-1} \\ & & & a_n & b_n^j \end{pmatrix}, \quad d^j = \begin{pmatrix} d_1^j \\ d_2^j \\ \vdots \\ d_n^j \end{pmatrix}, \quad \text{and} \quad G^j = \begin{pmatrix} G_1^j \\ G_2^j \\ \vdots \\ G_n^j \end{pmatrix},$$

where a_i and c_i are constants, whereas b_i^j and d_i^j are changing by the index i and j defined by

$$\begin{aligned} a_i &= -\frac{\theta_1}{(\Delta x)^2} + \frac{\theta_2}{2\Delta x} \\ b_i^j &= \frac{1}{\Delta\tau} + \alpha_i^j + \frac{2\theta_1}{(\Delta x)^2} \\ c_i &= -\frac{\theta_1}{(\Delta x)^2} - \frac{\theta_2}{2\Delta x} \\ d_i^j &= \frac{\mathcal{V}_i^j}{\Delta\tau} + \big(\mathcal{J}^\epsilon \mathcal{V}^j\big)_i - \lambda_\epsilon \mathcal{V}_i^j + F_i^j. \end{aligned}$$

We have denoted above the notations: $\alpha_i^j := \alpha(\tau_j, x_i)$, $F_i^j := F(\tau_j, x_i)$ and $G_i^j := G(\tau_j, x_i)$.

When discretizing the differential operator $\mathcal{A}$ we used points on the boundary. This requires the vector d^j needs to be adjusted by updating the first and the last entries of d^j as follows:

$$d_1^j \leftarrow d_1^j - a_1 \times G_1^j, \quad d_{N_x-1}^j \leftarrow 0. \tag{4.11}$$

According to the Theorem 5.2 in Jaillet et al. [21] the linear complementarity problem (4.10) has a unique solution if and only if the matrix $\mathbf{T}$ is coercive in the sense that

$$\exists C > 0 \text{ such that } \forall x \in \mathbb{R}^n, \big(\mathbf{T}x, x\big) \geq C|x|^2,$$

where (u, v) denotes the inner product of two vectors $u, v \in \mathbb{R}^n$.

4.3 Brennan-Schwartz algorithm

To solve the discrete linear complementarity problem (4.10), we apply the Brennan-Schwartz algorithm [7] employed in [21] and [1] for the pricing of American put option. We refer to Section 5 in Jaillet et al. [21] for detailed discussion on the complete justification of the algorithm.

Following Proposition 5.4 of [21], the unique solution of the system (4.10) can be computed for each $j = 0, 1, \dots, N_t$ by solving the following recursive relations:

- Step 1: Compute recursively a vector $\widetilde{b}$ as

$$\begin{aligned} \widetilde{b}_n &= b_n^{j+1} \\ \widetilde{b}_i &= b_{i-1}^{j+1} - c_{i-1} a_i / \widetilde{b}_i, \quad i = n, \dots, 2 \end{aligned}$$

- Step 2: Compute recursively a vector $\widetilde{d}$ as

$$\begin{aligned} \widetilde{d}_n &= d_n \\ \widetilde{d}_{i-1} &= d_{i-1} - c_{i-1} \widetilde{d}_i / \widetilde{b}_i, \quad i = n, \dots, 2 \end{aligned}$$

- Step 3: Compute vector $\mathcal{V}^{j+1}$ forward as follows

$$\begin{aligned} \mathcal{V}_1^{j+1} &= \max\big\{\widetilde{d}_1 / b_1^{j+1}, G_1^{j+1}\big\} \\ \mathcal{V}_i^{j+1} &= \max\big\{\big(\widetilde{d}_i - a_i \mathcal{V}_{i-1}^{j+1}\big)/\widetilde{b}_i, G_i^{j+1}\big\}, \quad i = 2, \dots, n. \end{aligned}$$

5 Numerical examples

To illustrate the problem (2.5), consider an equity-maximizing firm whose management acts in the best interest of equityholder. The firm is partly financed by debt with principal amount of P. The debt pays in exchange to the investor streams of payments paid continuously prior to and at default. A portion of each debt payment made is applied towards paying the interest (coupon) on the debt at a fixed rate $\rho > 0$. Another portion of the payment is applied towards reducing the principal by an amount p according to certain maturity profile $m \geq 0$, such that

$$P = \int_0^T e^{-mt} p dt, \quad \text{or equivalently,} \quad p = \frac{mP}{\big(1 - e^{-mT}\big)},$$

where we set $p = \frac{P}{T}$ for $m = 0$, meaning that the debt repayment is done uniformly in time.

The value of firm's asset is assumed to be independent of capital structure choices of the firm and is governed under risk-neutral measure $\mathbb{P}_{t,x}$ by the exponential Lévy process (2.3).

We assume that there is a corporate tax rate $\gamma > 0$ which depends on the firm value in the following way. As introduced in Leland and Toft [26] (also Hilberink and Rogers [18]) we assume that there exists a tax cutoff level V_T whose effect is that the tax rebates are $-\varphi X_t$, with $\varphi \in (0,1)$, while $X_t < V_T$, (contrarily to [26] with $\varphi = 0$) and are equal to $\gamma\rho P$ when the firm value $X_t \geq V_T$. When default happens at time τ, a fraction η of the firm asset is deducted for bankruptcy cost. Under these assumptions, the total value of the firm becomes

$$\begin{aligned}\mathcal{V}(t,x) =& \mathbb{E}_{t,x}\Big(\int_t^{\tau\wedge T} e^{-r(u-t)}\big[\delta X_u + \gamma\rho P \mathbf{1}_{\{X_u > V_T\}} - \varphi X_u \mathbf{1}_{\{X_u \leq V_T\}}\big] du\Big) \\ & - \eta \mathbb{E}_{t,x}\Big(e^{-r\tau} X_\tau \mathbf{1}_{\{\tau < T\}}\Big) + \mathbb{E}_{t,x}\Big(e^{-r(T-t)} X_T \mathbf{1}_{\{\tau \geq T\}}\Big).\end{aligned}$$

The total value of debt for the firm is given by

$$\begin{aligned}\mathcal{D}(t,x) =& \mathbb{E}_{t,x}\Big(\int_t^{\tau\wedge T} e^{-(r+m)(u-t)}\big[\rho P + p\big] du\Big) + (1-\eta)\mathbb{E}_{t,x}\Big(e^{-(r+m)(\tau-t)} X_\tau \mathbf{1}_{\{\tau < T\}}\Big) \\ & + \mathbb{E}_{t,x}\Big(e^{-(r+m)(T-t)} p \mathbf{1}_{\{\tau \geq T\}}\Big),\end{aligned}$$

where the first term accounts for interest payment and principal repayment of the debt until maturity time T or default time τ, whichever is sooner. The second term represents the remains of firm's value after bankruptcy cost is deducted. The last term is the final repayment of the debt principal when default doesn't occur. As the management acts in the best interest of the equityholders, the value of the firm's equity is determined by the optimal stopping problem

$$\begin{aligned}\mathcal{E}(t,x) = \sup_{\tau \in \mathcal{T}_{[t,T]}} & \mathbb{E}_{t,x}\Big(\int_t^{\tau\wedge T} e^{-r(u-t)}\big[\delta X_u + \gamma\rho P \mathbf{1}_{\{X_u > V_T\}} - \varphi X_u \mathbf{1}_{\{X_u \leq V_T\}} - e^{-m(u-t)}(\rho P + p)\big] du \\ & + e^{-r(\tau-t)}\big[-\eta + e^{-m(\tau-t)}(\eta - 1)\big] X_\tau \mathbf{1}_{\{\tau < T\}} + e^{-r(T-t)}\big[X_T - e^{-m(T-t)} p\big]\mathbf{1}_{\{\tau \geq T\}}\Big).\end{aligned}$$

The functions F, G, H and α of the optimal stopping problem are identified as follows:

$$\begin{aligned}F(t,x) &= \delta x + \gamma\rho P \mathbf{1}_{\{x > V_T\}} - \varphi x \mathbf{1}_{\{x \leq V_T\}} - e^{-mt}\big(\rho P + p\big), \\ G(t,x) &= \big[-\eta + e^{-mt}(\eta - 1)\big] x, \\ H(x) &= x - e^{-mT} p, \\ \alpha(t,x) &= r.\end{aligned}$$

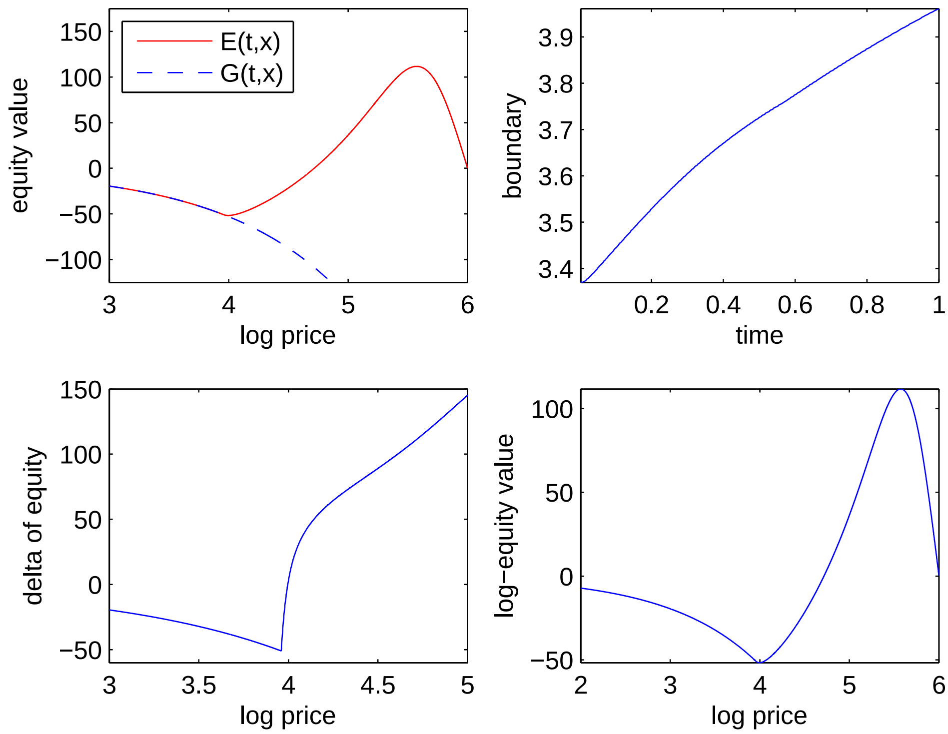

Figure 1: Firm equity value as a function of log-price (x) and the optimal boundary as a function of calendar time (t). In this case the firm asset (2.3) includes diffusion with $\sigma = 0.2$.

Note that the stopping cost function G posses the property of Assumption 3.5.

For all computations, we fix some values of model parameters. We set debt principal $P = 60$ with $T = 1$ and maturity profile $m = 5\%$; interest rate $r = 10\%$; payout rate $\delta = 7\%$; tax benefit and payment rates $\gamma = 35\%$ and $\varphi = 10\%$, respectively; coupon rate $\rho = 5\%$; bankruptcy cost $\eta = 40\%$. The jump structure of the underlying asset process (2.3) is governed by two-sided jumps tempered stable Lévy process (4.4) with parameter values set: $C_1 = 2$, $C_2 = 1$, $G = 7$, $M = 9$ and $Y = 0.1$. The numerical results are presented in Figure 1, Figure 2 and Figure 3.

We observe from Figure 1 that the equity value $\mathcal{E}$ of the firm joints smoothly on the boundary with the stopping cost function G. In contrast to this, we observe in Figure 2 that such smooth pasting doesn't hold as there is a jump in the derivative (delta) of the firm equity value on the boundary. This observation is found to be consistent with the recent result in [25] for perpetual case. In both figures we see that the firms's equity value function $\mathcal{E}(t, x)$ dominates the stopping cost function $G(t, x)$ for all values of t and x and the optimal boundary is decreasing in time to maturity. Figure 3 verifies that the boundary and the value function (the dash line on the surface) are increasing and decreasing in calendar time (t), respectively.

A Proof of main results

A.1 Proof of Theorem 3.1

Proof As the function W is $\mathcal{C}^1$ and twice $\mathcal{C}^2$ continuously differentiable in t and x, respectively,

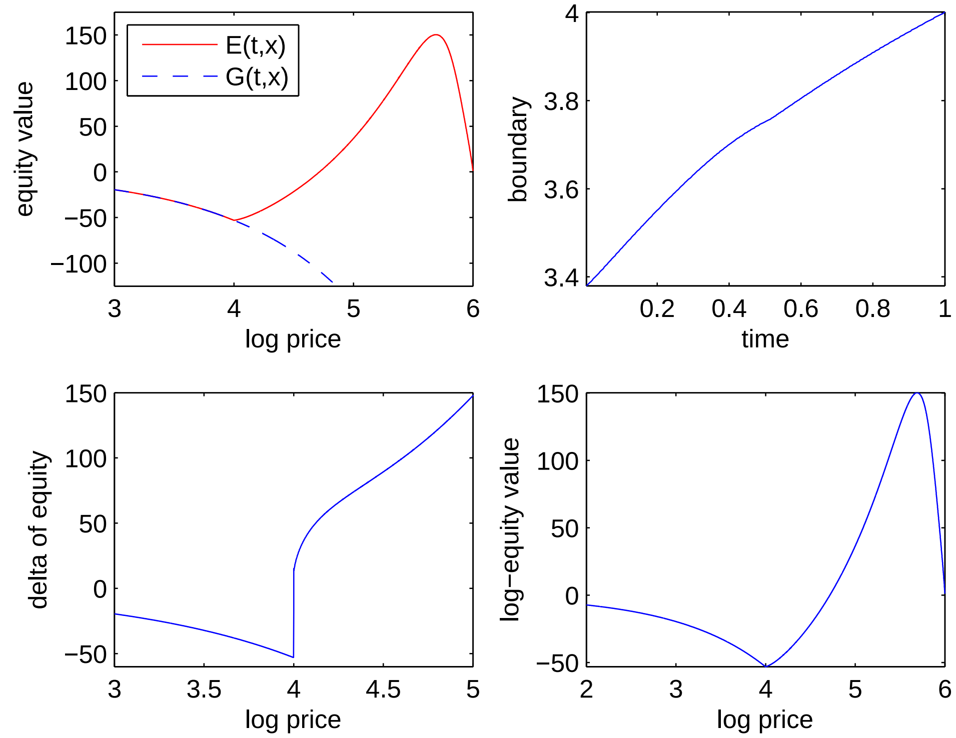

Figure 2: Firm equity value as a function of log-price (x) and the optimal boundary as a function of calendar time (t). In this case the firm asset (2.3) excludes diffusion, i.e., $\sigma = 0$.

according to Itô's formula (see e.g., Theorem II.33 in Protter [33] or formula (4.58) in Jacod and Shiryaev [20]) applied to $e^{-\int_t^s \alpha(\theta,X_\theta)d\theta}W(s,X_s)$ with $s \geq t$, we have that

$$\begin{aligned} e^{-\int_t^s \alpha(\theta,X_\theta)d\theta}W(s,X_s) + \int_t^s e^{-\int_t^u \alpha(\theta,X_\theta)d\theta}F(u,X_u)du = W(t,x) \\ -\int_t^s e^{-\int_t^u \alpha(\theta,X_\theta)d\theta}\Big[-\frac{\partial W}{\partial t} - \mathcal{L}W - F\Big](u,X_u)du + \mathcal{M}_s, \end{aligned} \tag{A.1}$$

where $\mathcal{M}$ is part of the Itô's decomposition taking the following form

$$\begin{aligned} \mathcal{M}_s =& \sigma \int_t^s e^{-\int_t^u \alpha(\theta,X_\theta)d\theta} X_u \frac{\partial W}{\partial x}(u,X_u)dB_u \\ &+ \int_t^s \int_{\mathbb{R}} e^{-\int_t^u \alpha(\theta,X_\theta)d\theta}\big\{W(u,X_u e^y) - W(u,X_u)\big\}\widehat{\nu}(dy,du). \end{aligned} \tag{A.2}$$

Let us show that $\mathcal{M}$ is a martingale. To deal with this, note that since W is $C^{1,2}$ we have

$$\begin{aligned} \mathbb{E}_{t,x}\Big(\int_t^s \int_{\mathbb{R}} \big|W(u,X_u e^y) - W(u,X_u)\big|^2 \Pi(dy)du\Big) \leq& \mathbb{E}_{t,x}\Big(\int_t^s \int_{\mathbb{R}} c^2\big|e^y - 1\big|^2 X_u^2 \Pi(dy)du\Big) \\ \leq& C_1^2 \int_{\mathbb{R}} \big|e^y - 1\big|^2 \Pi(dy)\mathbb{E}_{t,x}\Big(\int_t^s X_u^2 du\Big) \\ <& \infty, \end{aligned}$$

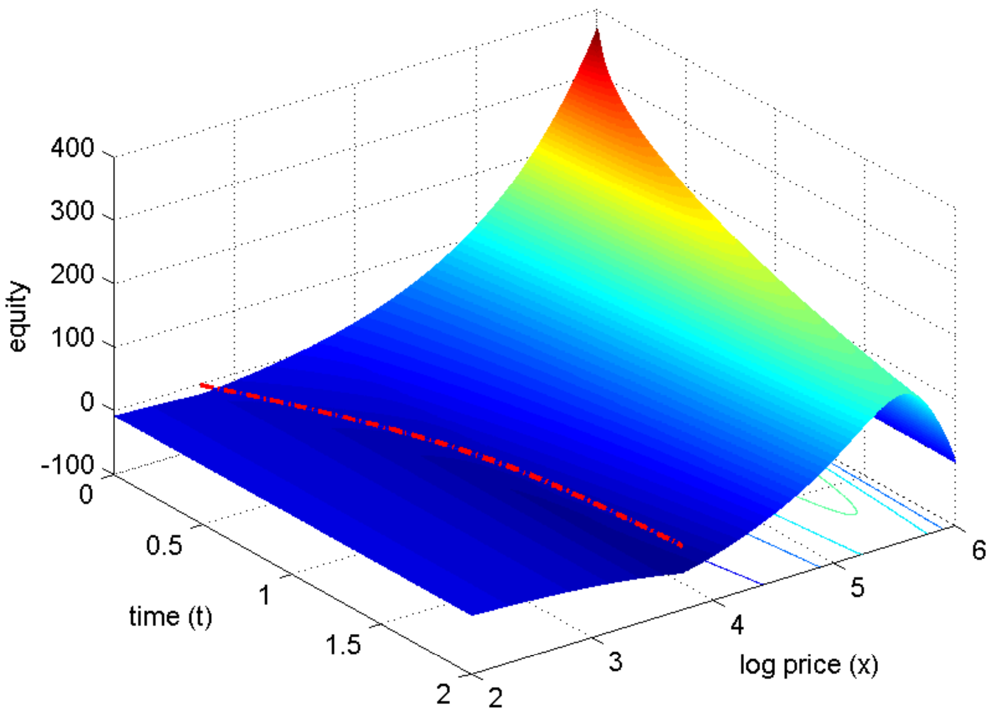

Figure 3: Firm equity value $\mathcal{E}(t,x)$ as a function of log-price (x) and calendar time (t). The dash line on the surface is the optimal boundary.

where the last inequality is due to the integrability conditions (2.2) and (2.4). Using the compensation formula (see for instance Theorem 4.4 in Kyprianou [24]) we know that

$$\mathbb{E}_{t,x}\Big(\int_t^s \int_{\mathbb{R}} \big|W(u, X_u e^y) - W(u, X_u)\big|^2 \nu(dy, du)\Big)$$
$$= \mathbb{E}_{t,x}\Big(\int_t^s \int_{\mathbb{R}} \big|W(u, X_u e^y) - W(u, X_u)\big|^2 \mu(dy, du)\Big).$$

Hence, the second term in (A.2) is a square $\mathbb{P}_{t,x}-$martingale. Also, since W is $C^{1,2}$,

$$\mathbb{E}_{t,x}\Big(\int_t^s X_u^2 \Big(\frac{\partial W}{\partial x}\Big)^2 (u, X_u) du\Big) \leq C_2 \mathbb{E}_{t,x}\Big(\int_t^s X_u^2 du\Big) < \infty,$$

where both C_1 and C_2 are positive constants. Thus, using the Itô's isometry theorem we obtain that the first term in (A.2) is a square integrable $\mathbb{P}_{t,x}-$martingale, and therefore so is the process $\mathcal{M}$ (A.2). As a result, we have that $\mathbb{E}_{t,x}(\mathcal{M}_s) = 0$ for each $t \leq s \leq T$.

Next, replace s in (A.1) by $\tau \wedge T := \min\{\tau, T\}$, where $\tau \in \mathcal{T}_{[t,T]}$. By using the conditions

(3.6), (3.4) and (3.5), we have after taking $\mathbb{P}_{t,x}-$expectation a series of inequalities:

$$\begin{aligned}
&\mathbb{E}_{t,x}\Big(e^{-\int_t^T \alpha(s,X_s)ds}H(X_T)\mathbf{1}_{\{\tau\geq T\}} + e^{-\int_t^\tau \alpha(s,X_s)ds}G(\tau,X_\tau)\mathbf{1}_{\{\tau<T\}}\\
&\qquad\qquad\qquad + \int_t^{\tau\wedge T} e^{-\int_t^u \alpha(s,X_s)ds}F(u,X_u)du\Big)\\
&\leq \mathbb{E}_{t,x}\Big(e^{-\int_t^T \alpha(s,X_s)ds}W(T,X_T)\mathbf{1}_{\{\tau\geq T\}} + e^{-\int_t^\tau \alpha(s,X_s)ds}W(\tau,X_\tau)\mathbf{1}_{\{\tau<T\}}\\
&\qquad\qquad\qquad + \int_t^{\tau\wedge T} e^{-\int_t^u \alpha(s,X_s)ds}F(u,X_u)du\Big)\\
&= \mathbb{E}_{t,x}\Big(e^{-\int_t^{\tau\wedge T} \alpha(s,X_s)ds}W(\tau\wedge T,X_{\tau\wedge T}) + \int_t^{\tau\wedge T} e^{-\int_t^u \alpha(s,X_s)ds}F(u,X_u)du\Big)\\
&= W(t,x) - \mathbb{E}_{t,x}\Big(\int_t^{\tau\wedge T} e^{-\int_t^u \alpha(s,X_s)ds}\big[-\frac{\partial W}{\partial t} - \mathcal{L}W - F\big](u,X_u)du\Big)\\
&\leq W(t,x).
\end{aligned}$$

Since τ is any stopping time in $\mathcal{T}_{[t,T]}$, we conclude from the above series of inequalities that

$$V(t,x)\leq W(t,x). \tag{A.3}$$

To get the other inequality, replace s in (A.1) by $\tau_b^-\wedge T$, where τ_b^- is defined in (3.8). Using the facts that $-\frac{\partial W}{\partial t}(t,x) - \mathcal{L}W(t,x) - F(t,x) = 0$ for every $t\in(0,T)$ and $x>b(t)$, $W(T,x)=H(x)$ for all $x\in\mathbb{R}_+$ and $W(t,x)=G(t,x)$ for every $t\in(0,T), x\leq b(t)$, followed from the conditions (3.2), (3.6) and (3.3), we have after taking $\mathbb{P}_{t,x}-$expectation that

$$\begin{aligned}
W(t,x) &= \mathbb{E}_{t,x}\Big(e^{-\int_t^{\tau_b^-\wedge T} \alpha(s,X_s)ds}W(\tau_b^-\wedge T,X_{\tau_b^-\wedge T}) + \int_t^{\tau_b^-\wedge T} e^{-\int_t^u \alpha(s,X_s)ds}F(u,X_u)du\Big)\\
&= \mathbb{E}_{t,x}\Big(e^{-\int_t^T \alpha(s,X_s)ds}W(T,X_T)\mathbf{1}_{\{\tau_b^-\geq T\}} + e^{-\int_t^{\tau_b^-} \alpha(s,X_s)ds}W(\tau_b^-,X_{\tau_b^-})\mathbf{1}_{\{\tau_b^-<T\}}\\
&\qquad\qquad + \int_t^{\tau_b^-\wedge T} e^{-\int_t^u \alpha(s,X_s)ds}F(u,X_u)du\Big)\\
&= \mathbb{E}_{t,x}\Big(e^{-\int_t^T \alpha(s,X_s)ds}H(X_T)\mathbf{1}_{\{\tau_b^-\geq T\}} + e^{-\int_t^{\tau_b^-} \alpha(s,X_s)ds}G(\tau_b^-,X_{\tau_b^-})\mathbf{1}_{\{\tau_b^-<T\}} \qquad (A.4)\\
&\qquad\qquad + \int_t^{\tau_b^-\wedge T} e^{-\int_t^u \alpha(s,X_s)ds}F(u,X_u)du\Big)\\
&\leq \sup_{\tau\in\mathcal{T}_{[t,T]}}\mathbb{E}_{t,x}\Big(e^{-\int_t^T \alpha(s,X_s)ds}H(X_T)\mathbf{1}_{\{\tau\geq T\}} + e^{-\int_t^\tau \alpha(s,X_s)ds}G(\tau,X_\tau)\mathbf{1}_{\{\tau<T\}}\\
&\qquad\qquad + \int_t^{\tau\wedge T} e^{-\int_t^u \alpha(s,X_s)ds}F(u,X_u)du\Big)\\
&= V(t,x),
\end{aligned}$$

from which we deduce following (A.3) that

$$\begin{aligned}
W(t,x) = \mathbb{E}_{t,x}\Big(&\int_t^{\tau_b^-\wedge T} e^{-\int_t^u \alpha(s,X_s)ds}F(u,X_u)du\\
&+ e^{-\int_t^{\tau_b^-} \alpha(s,X_s)ds}G(\tau_b^-,X_{\tau_b^-})\mathbf{1}_{\{\tau_b^-<T\}} + e^{-\int_t^T \alpha(s,X_s)ds}H(X_T)\mathbf{1}_{\{\tau_b^-\geq T\}}\Big) = V(t,x),
\end{aligned}$$

showing that the value function V of the optimal stopping problem (2.5) coincides with the function W that solves the free-boundary problem (3.2)-(3.7).

The proof for the boundary b solving (3.9) is as follows. Replace $s = T$ in (A.1). Taking account of the conditions (3.6), (3.2) and (3.3) we have after taking $\mathbb{P}_{t,x}-$expectation that

$$\begin{aligned}&V(t,x) - \mathbb{E}_{t,x}\Big(\int_t^T e^{-\int_t^u \alpha(s,X_s)ds}\big[-\frac{\partial G}{\partial t} - \mathcal{L}G - F\big](u,X_u)\mathbf{1}_{\{X_u \le b(u)\}}du\Big)\\ =&\mathbb{E}_{t,x}\Big(e^{-\int_t^T \alpha(s,X_s)ds}H(X_T) + \int_t^T e^{-\int_t^u \alpha(s,X_s)ds}F(u,X_u)du\Big).\end{aligned} \tag{A.5}$$

The proof is complete after inserting $x = b(t)$ in the above expression and again use (3.3). □

A.2 Proof of Theorem 3.2

Suppose that (W, c) be a solution pair to the free-boundary problem (3.2)-(3.7). Define

$$\tau_c^- = \inf\{t \le s \le T : X_s < c(s)\}.$$

Applying Itô's change-of-variable formula to $e^{-\int_t^s \alpha(\theta,X_\theta)d\theta}W(s,X_s)$, for $s \ge t$, we have

$$\begin{aligned}&e^{-\int_t^s \alpha(\theta,X_\theta)d\theta}W(s,X_s) + \int_t^s e^{-\int_t^u \alpha(\theta,X_\theta)d\theta}F(u,X_u)du = W(t,x)\\ &\qquad - \int_t^s e^{-\int_t^u \alpha(\theta,X_\theta)d\theta}\big[-\frac{\partial G}{\partial t} - \mathcal{L}G - F\big](u,X_u)\mathbf{1}_{\{X_u \le c(u)\}}du + \mathcal{M}_s,\end{aligned} \tag{A.6}$$

where $\mathcal{M}$ is a square-integrable $\mathbb{P}_{t,x}-$martingale (A.2). Recall that $W(T,x) = H(x)$ for all $x \in \mathbb{R}_+$ and $W(t,x) = G(t,x)$ for every $t \in (0,T), x \le c(t)$. Following these two facts, one can deduce following the same arguments as before that $c(t)$ solves the integral equation (3.9) and

$$\begin{aligned}W(t,x) \;=\; \mathbb{E}_{t,x}\Big(&e^{-\int_t^T \alpha(s,X_s)ds}H(X_T)\mathbf{1}_{\{\tau_c^- \ge T\}} + e^{-\int_t^{\tau_c^-} \alpha(s,X_s)ds}G(\tau_c^-, X_{\tau_c^-})\mathbf{1}_{\{\tau_c^- < T\}}\\ &+ \int_t^{\tau_c^- \wedge T} e^{-\int_t^u \alpha(s,X_s)ds}F(u,X_u)du\Big).\end{aligned}$$

Note that the value function V of the problem (2.5) solves the same free-boundary problem as the function W does with the optimal stopping time τ_b^- being the first exit of the price process X below a boundary b that solves the same integral equation (3.9) as $c(t)$ does, i.e.,

$$\begin{aligned}V(t,x) \;=\; \mathbb{E}_{t,x}\Big(&\int_t^{\tau_b^- \wedge T} e^{-\int_t^u \alpha(s,X_s)ds}F(u,X_u)du + e^{-\int_t^{\tau_b^-} \alpha(s,X_s)ds}G(\tau_b^-, X_{\tau_b^-})\mathbf{1}_{\{\tau_b^- < T\}}\\ &+ e^{-\int_t^T \alpha(s,X_s)ds}H(X_T)\mathbf{1}_{\{\tau_b^- \ge T\}}\Big).\end{aligned}$$

Following (A.4), we have for every $t \in [0,T]$ and $x \in \mathbb{R}_+$ that

$$V(t,x) \ge W(t,x),$$

which in turn implies that

$$c(t) \ge b(t) \quad \text{for all } t \in [0,T].$$

This inequality for the two boundaries also gives that

$$\tau_b^- \ge \tau_c^- \quad \text{almost surely.}$$

Now suppose that there exists $t \in (0,T)$ such that $c(t) > b(t)$. Next, let us take for a given $t \in (0,T)$ a point $x \in (b(t), c(t))$. By replacing s with stopping time $\tau_b^- \wedge T$ in (A.6), we obtain

$$\begin{aligned} V(t,x) &= \mathbb{E}_{t,x}\Big(e^{-\int_t^{\tau_b^-}\alpha(s,X_s)ds}G(\tau_b^-, X_{\tau_b^-})\mathbf{1}_{\{\tau_b^-<T\}} + e^{-\int_t^T \alpha(s,X_s)ds}H(X_T)\mathbf{1}_{\{\tau_b^-\geq T\}} \\ &\qquad + \int_t^{\tau_b^-\wedge T} e^{-\int_t^u \alpha(s,X_s)ds}F(u,X_u)du\Big) \\ &= \mathbb{E}_{t,x}\Big(e^{-\int_t^{\tau_b^-}\alpha(s,X_s)ds}W(\tau_b^-, X_{\tau_b^-})\mathbf{1}_{\{\tau_b^-<T\}} + e^{-\int_t^T \alpha(s,X_s)ds}W(T,X_T)\mathbf{1}_{\{\tau_b^-\geq T\}} \\ &\qquad + \int_t^{\tau_b^-\wedge T} e^{-\int_t^u \alpha(s,X_s)ds}F(u,X_u)du\Big) \\ &= \mathbb{E}_{t,x}\Big(e^{-\int_t^{\tau_b^-\wedge T}\alpha(s,X_s)ds}W(\tau_b^-\wedge T, X_{\tau_b^-\wedge T}) + \int_t^{\tau_b^-\wedge T} e^{-\int_t^u \alpha(s,X_s)ds}F(u,X_u)du\Big) \\ &= W(t,x) - \mathbb{E}_{t,x}\Big(\int_t^{\tau_b^-\wedge T} e^{-\int_t^u \alpha(s,X_s)ds}\big[-\frac{\partial G}{\partial t} - \mathcal{L}G - F\big](u,X_u)\mathbf{1}_{\{X_u\leq c(u)\}}du\Big), \end{aligned}$$

where we have used in the second equality the fact that $X_{\tau_b^-} < b(\tau_b^-)$, $W(T,x) = H(x)$ for all $x \in \mathbb{R}_+$, $W(t,x) = G(t,x)$ for every $t \in (0,T)$ and $x < c(t)$ and the fact that $c(t) > b(t)$. On remarking that $V(t,x) \geq W(t,x)$ for every $(t,x) \in [0,T) \times \mathbb{R}_+$, we deduce that

$$\mathbb{E}_{t,x}\Big(\int_t^{\tau_b^-\wedge T} e^{-\int_t^u \alpha(s,X_s)ds}\big[-\frac{\partial G}{\partial t} - \mathcal{L}G - F\big](u,X_u)\mathbf{1}_{\{X_u\leq c(u)\}}du\Big) \leq 0,$$

which can not be true, due to the imposed condition (3.10) of the theorem. Hence, as a result in the absence of the existence of such a point x, it follows that

$$b(t) = c(t) \quad \text{for all } t \in [0,T],$$

which completes the proof on the uniqueness of solution to the integral equation (3.9). □

A.3 Proof of Lemma 3.1

The first part of the proof follows from (A.5). To show that U (2.6) solves the problem (4.2), we use once again the Itô's formula applied to $e^{-\int_t^s \alpha(\theta,X_\theta)d\theta}W(s,X_s)$ for $s \geq t$ to obtain

$$\begin{aligned} e^{-\int_t^s \alpha(\theta,X_\theta)d\theta}U(s,X_s) + \int_t^s e^{-\int_t^u \alpha(\theta,X_\theta)d\theta}F(u,X_u)du = U(t,x) \\ - \int_t^s e^{-\int_t^u \alpha(\theta,X_\theta)d\theta}\big[-\frac{\partial U}{\partial t} - \mathcal{L}U - F\big](u,X_u)du + \mathcal{M}_s, \end{aligned} \tag{A.7}$$

where $\mathcal{M}$ is a square integrable $\mathbb{P}_{t,x}-$martingale process. Taking account that $U(T,x) = H(x)$ $\forall x \in \mathbb{R}_+$ and that $-\frac{\partial U}{\partial t} - \mathcal{L}U - F = 0$ $\forall (t,x) \in [0,T) \times \mathbb{R}_+$, the proof that U solves the problem (3.11)-(3.12) is complete after taking the $\mathbb{P}_{t,x}-$expectation on both side of (A.7). □

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